Econ 802

Answers to Final Exam

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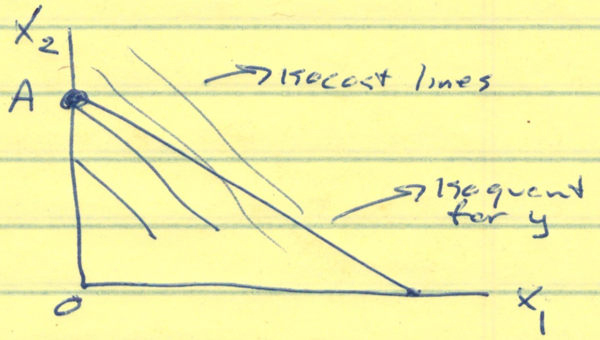
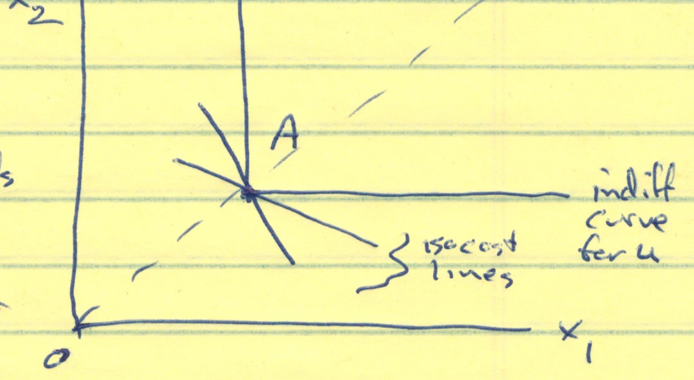
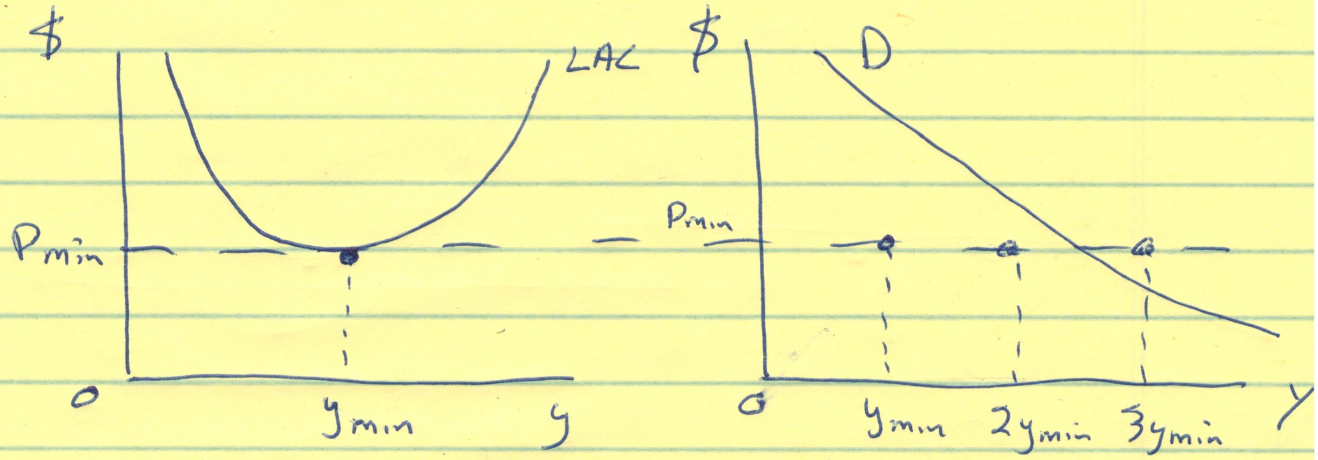
1. (a) Consider the linear production function $y = ax_1 + bx_2$ where $a > 0$, $b > 0$. The marginal products are a and b, which are both positive. However, a cost-minimizing firm will typically operate at a boundary solution where it only uses one input. For example if the firm wants to produce y and the isocost lines are steeper than the isoquant, it will operate at point A where $x_1 = 0$.



(b) Consider the Leontief utility function $u = \min\{ax_1, bx_2\}$ where $a > 0$ and $b > 0$. Suppose the consumer minimizes the expenditure $p_1x_1 + p_2x_2$ for some given utility level u: Regardless of the prices, the expenditure-minimizing point is always the corner A. Therefore the Hicksian demands $h_1(p, u)$ and $h_2(p, u)$ must be constants that are independent of the price vector p, so the Hicksian demand curves cannot be downward sloping.



1(c) Suppose firms have U-shaped LAC curves. For example the cost function $c(y) = A + By^2$ with $A > 0$, $B > 0$ gives $LAC = \frac{A}{y} + By$. A little calculus shows that this is minimized at $y_{min} = \left(\frac{A}{B}\right)^{1/2}$ and the resulting LAC is $2(AB)^{1/2}$. Call this p_{min}.



If $p > p_{min}$, firms have positive profit. Because firms are price takers, new firms enter and supply > demand. If $p < p_{min}$ then any firm with positive output has negative profit, so all firms exit and supply < demand. If $p = p_{min}$, no firm is willing to produce a fractional output $0 < y < y_{min}$ because this would give negative profit, so supply cannot equal demand at this price either.

2(a) The easiest way to think about this is to notice that $g(w,e)$ is mathematically identical to an indirect utility function $v(p,m)$. Just replace $u(x)$ by $f(x)$, p by w, and m by e. So $g(w,e)$ will have all of the same mathematical properties as $v(p,m)$; see Varian pp. 102–103.

Properties of $g(w,e)$

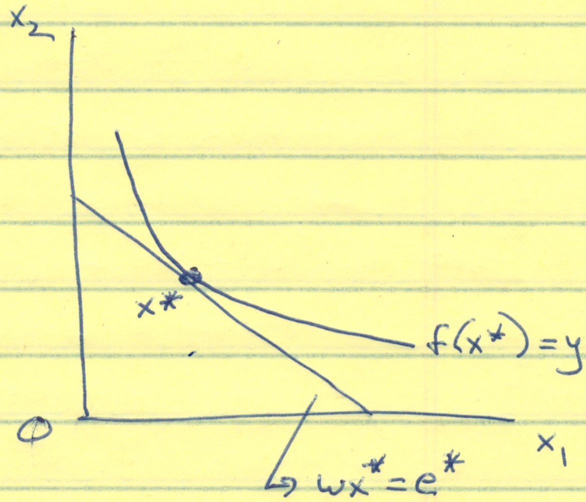
(i) Nonincreasing in w and nondecreasing in e.
This follows from the fact that an increase in a price makes the feasible set smaller, and an increase in cost (expenditure) makes the feasible set larger.

(ii) Homogeneous of degree zero in (w,e). This follows from the fact that the feasible set is identical for (w,e) and (tw,te) where $t>0$. Therefore the solution x^* is identical and the maximum output $f(x^*)$ is identical.

(iii) Quasi-convexity in w. I won't write out the proof but it is identical to Varian p. 103.

(iv) Continuous for $(w,e)>0$ due to the Theorem of the Maximum.

2.(b) Consider the following graph:



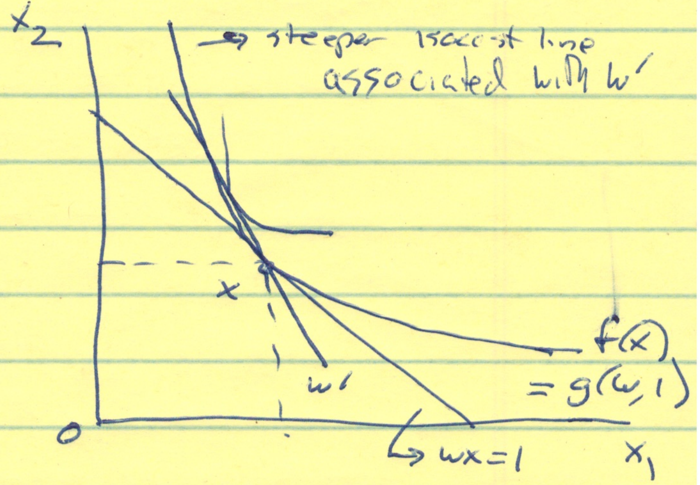
For given prices w, if the firm minimizes the cost of producing y^*, the optimal input bundle is x^* and the resulting cost is $wx^*=e^*$. So $e^*=c(w,y^*)$. By duality, if the firm maximizes the output it can achieve for the expenditure e^*, the optimal bundle is again x^* and the resulting output is y^* so $y^*=g(w,e^*)$. More generally, for any number of inputs, the two functions are inverses. If you know $c(w,y)$ you can write $e=c(w,y)$ and solve for y to get $y=g(w,e)$. If you know $g(w,e)$ you can write $y=g(w,e)$ and solve for e to get $e=c(w,y)$.

2(c) The analogy between $g(w,e)$ and the indirect utility function for a consumer $v(p,m)$ suggests using the following method:

$$f(x) = \min_{w} g(w,1) \text{ subject to } wx = 1.$$

(where $e \equiv 1$)

To see why this method will work, consider a graph for the 2 input case. Choose any $x > 0$. Then draw the isoquant through this point, set $e = 1$, and choose the prices w so that the resulting isocost line through x is tangent to the isoquant there. For the more general case we would choose $w = w(x)$ where $w(x)$ is the inverse demand function when $e = 1$. Clearly $f(x) = g(w,1)$, because this is the highest possible output at the prices w when $e = 1$.



Now consider any other price vector w' such that $w'x = 1$ so the bundle x is still affordable (the new isocost line passes through x with $e = 1$). This means $f(x)$ is still feasible, but in general it is also possible to get to a higher isoquant (see the steeper isocost line in the graph). Therefore $f(x) = g(w,1) \leq g(w',1)$ when $w'x = 1$. This implies

$$f(x) = \min_{w} g(w,1) \text{ subject to } wx = 1$$

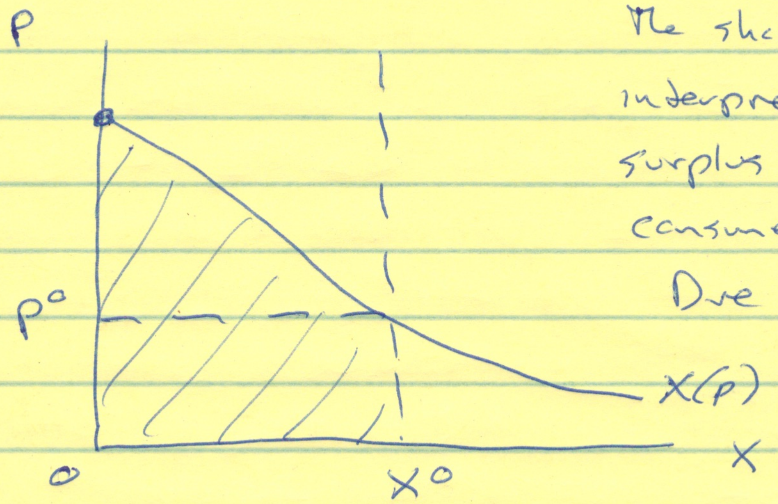
so the method works.

3(a) The Marshallian demand comes from max $1-e^{-rx}+y$ subject to $px+y=w$. By substitution we choose x to max $1-e^{-rx}+w-px$. FOC: $re^{-rx}-p=0$
SOC: $-r^2e^{-rx}<0$ which is a sufficient condition for a max.
Marshallian demand is $x(p) = -\frac{1}{r}\ln\frac{p}{r}$
[Note: This only makes sense when $p \le r$ so $x(p) \ge 0$.]
The Hicksian demand comes from min $px+y$ subject to $1-e^{-rx}+y=u$. By substitution we choose x to min $px+u-1+e^{-rx}$. FOC: $p-re^{-rx}=0$
SOC: $r^2e^{-rx}>0$ which is a sufficient condition for a min.
Hicksian demand is $h(p) = -\frac{1}{r}\ln\frac{p}{r}$
[So the Hicksian and Marshallian demands are identical; This is due to quasi-linear utility and the absence of income effects]

(b) To obtain $v(p,w)$, substitute the Marshallian demand into $1-e^{-rx}+w-px$ to get $v(p,w) = 1+w+\left(\frac{p}{r}\right)\left[\ln\left(\frac{p}{r}\right)-1\right]$
or for an individual Kamala,
$v_i(p,w_i) = 1+w_i+\left(\frac{p}{r_i}\right)\left[\ln\left(\frac{p}{r_i}\right)-1\right] \quad i=1\cdots n$
This is in the Gorman form ~~where~~ $a_i(p)+b(p)w_i$
where $a_i(p) = 1+\left(\frac{p}{r_i}\right)\left[\ln\left(\frac{p}{r_i}\right)-1\right]$ $b(p)\equiv 1$ and w_i is income. Therefore we can aggregate to get the indirect utility function $V(p,W) = \Sigma a_i(p) + W$ where $W = \Sigma w_i$ for one big Kamala. The aggregate demand $X(p)$ can be obtained from $V(p,W)$ using Roy's identity.

[Note: aggregation works here due to quasi-linearity.]

3(c) Consider the area under the market demand curve:



The shaded area can be interpreted as the total surplus from X^0 for all consumers as a group. Due to quasi-linearity and the absence of income effects, this is also the total gain in utility if X^0 is made available (and distributed efficiently among consumers so they all have the same marginal utility, which will be true if they all face the same price p^0). If the total gain in utility exceeds F then the net gain to the economy is positive. If the revenue $p^0X^0 > F$ then the government can refund $p^0X^0 - F$ units of the y good back to consumers. If $p^0X^0 < F$ then the government will have to collect some additional units of the y good from consumers to pay for ~~[illegible]~~ X^0.

4(a) In the short run, $y^2 = x_1 x_2$ or $x_1 = \frac{y^2}{x_2}$ where x_2 is fixed. The cost function is

$$c(y, x_2) = \frac{y^2}{x_2} + x_2 \quad \text{because } w_1 = w_2 = 1.$$

The firm maxes $py - \frac{y^2}{x_2} - x_2$

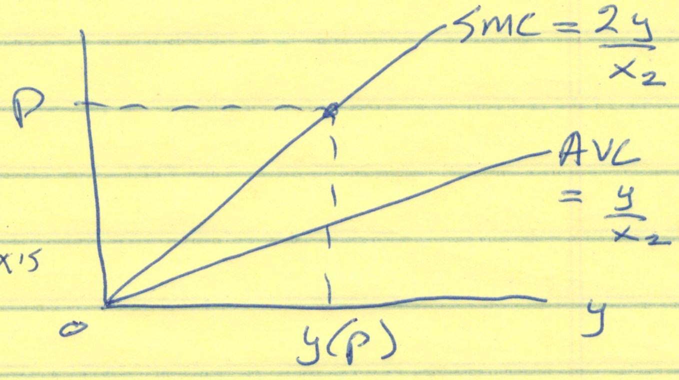
FOC: $p - \frac{2y}{x_2} = 0 \Rightarrow y(p) = \frac{px_2}{2}$

where sufficient SOC is satisfied because MC is rising.

The firm never shuts down because we always have

$P = SMC > AVC$.

The supply function $y(p)$ is identical to SMC except that we read across from the vertical axis to SMC and then down.

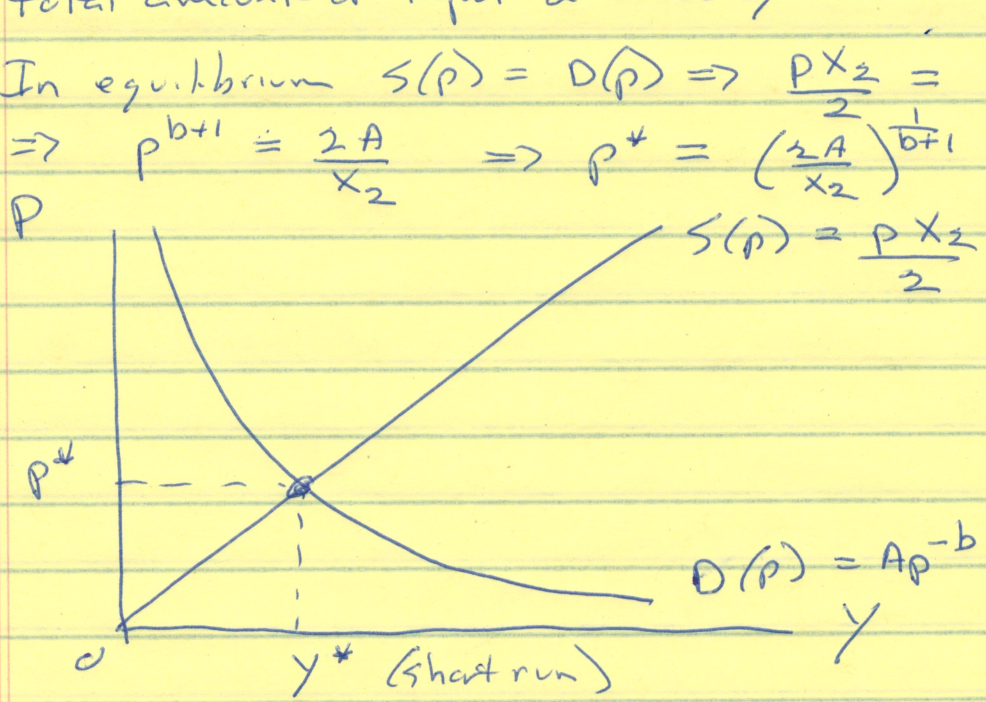


(b) We have $y_j(p) = \frac{p x_{j2}}{2}$ so $S(p) = \sum_j y_j(p)$

or $S(p) = \frac{p X_2}{2}$ where $X_2 = \sum_j x_{j2}$ is the total amount of input 2 used by all firms.

In equilibrium $S(p) = D(p) \Rightarrow \frac{p X_2}{2} = Ap^{-b}$

$\Rightarrow p^{b+1} = \frac{2A}{X_2} \Rightarrow p^* = \left(\frac{2A}{X_2}\right)^{\frac{1}{b+1}}$



4(c) First minimize cost in the long run. The Lagrangean is

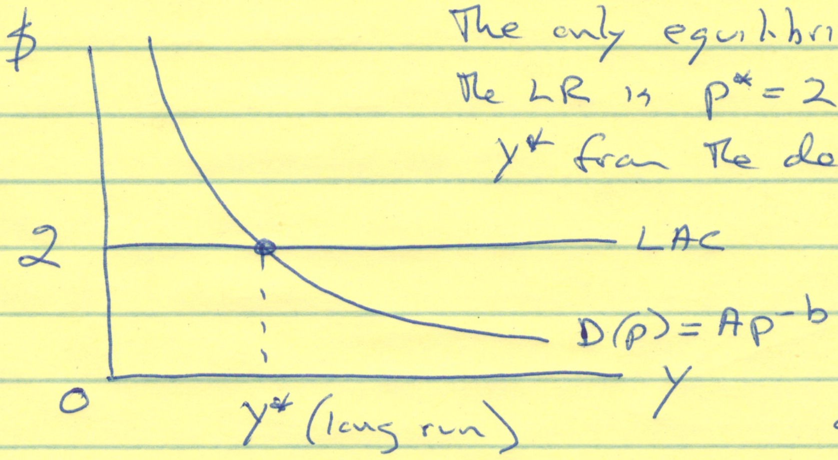
$$L = x_1 + x_2 - \lambda(x_1^{1/2} x_2^{1/2} - y) \quad \text{using } w_1 = w_2 = 1$$

$$\text{FOC: } \left.\begin{aligned} 1 - \lambda\left(\tfrac{1}{2}\right)x_1^{-1/2}x_2^{1/2} &= 0 \\ 1 - \lambda\left(\tfrac{1}{2}\right)x_1^{1/2}x_2^{-1/2} &= 0 \end{aligned}\right\} \Rightarrow 1 = \frac{x_2}{x_1}$$

So $x_1 = x_2 \Rightarrow y = x_1 = x_2$.

Therefore $c(y) = 2y \Rightarrow LAC = LMC = 2$

so we have a horizontal LAC (due to constant returns)



The only equilibrium we can have in the LR is $p^* = 2$ and the corresponding y^* from the demand function. $P > 2$ implies the profit max problems have no solution and $P < 2$ implies the firms produce zero.

Yes, price could be lower in the short run. This is true if $2 > \left(\frac{2A}{x_2}\right)^{\frac{1}{b+1}}$ or $x_2 > A2^{-b}$. In this case the aggregate amount of the fixed input is larger than what the firms would choose in the long run, so they produce a lot in the short run and drive down the price. This implies that in the short run they must have negative profit because price is below LAC, and we know $LAC \leq SAC$. But the firms do not shut down in the short run for the reasons discussed in part (a).

5(a) We will need the Marshallian demands for each person.

Write $L_A = \alpha \ln x_{A1} + (1-\alpha)\ln x_{A2} - \lambda_A\left[p_1 x_{A1} + p_2 x_{A2} - m_A\right]$

FOC: $\left.\begin{aligned} \frac{\alpha}{x_{A1}} &= \lambda_A p_1 \\ \frac{(1-\alpha)}{x_{A2}} &= \lambda_A p_2 \end{aligned}\right\} \Rightarrow \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{x_{A2}}{x_{A1}}\right) = \frac{p_1}{p_2}$

$$\Rightarrow p_2 x_{A2} = p_1 x_{A1}\left(\frac{1-\alpha}{\alpha}\right)$$

Note: This is a log transformation of a Cobb-Douglas utility function so it is strictly quasi-concave and we don't have to worry about SOC.

Substituting into the budget constraint gives

$$p_1 x_{A1} + p_1 x_{A1}\left(\frac{1-\alpha}{\alpha}\right) = m_A \quad \Rightarrow x_{A1}(p, m_A) = \frac{\alpha m_A}{p_1}$$

and $x_{A2}(p, m_A) = \dfrac{(1-\alpha)m_A}{p_2}$

A's income $m_A = p_1 w_{A1} + p_2 w_{A2}$ so

$$x_{A1} = \frac{\alpha}{p_1}\left[p_1 w_{A1} + p_2 w_{A2}\right] = \alpha\left[w_{A1} + \left(\frac{p_2}{p_1}\right) w_{A2}\right]$$

Going through the same process for B, we get the Marshallian demands $x_{B1}(p, m_B) = \dfrac{\beta m_B}{p_1}$

$$x_{B2}(p, m_B) = \frac{(1-\beta)m_B}{p_2}$$

where $m_B = p_1 w_{B1} + p_2 w_{B2}$ so

$$x_{B1} = \frac{\beta}{p_1}\left[p_1 w_{B1} + p_2 w_{B2}\right] = \beta\left[w_{B1} + \left(\frac{p_2}{p_1}\right) w_{B2}\right]$$

In equilibrium the market for good 1 must clear, so $x_{A1} + x_{B1} = W_1$ where $W_1 = w_{A1} + w_{B1}$

Therefore $\alpha\left[w_{A1} + \left(\frac{p_2}{p_1}\right) w_{A2}\right] + \beta\left[w_{B1} + \left(\frac{p_2}{p_1}\right) w_{B2}\right] = w_{A1} + w_{B1}$

$$\Rightarrow \left(\frac{p_2}{p_1}\right)\left[\alpha w_{A2} + \beta w_{B2}\right] = w_{A1} + w_{B1} - \alpha w_{A1} - \beta w_{B1}$$

$$\Rightarrow \frac{p_2}{p_1} = \frac{(1-\alpha) w_{A1} + (1-\beta) w_{B1}}{\alpha w_{A2} + \beta w_{B2}}$$

5(a) continued. Note that we only have to clear the market for one good, because Walras's Law implies that the other market clears automatically. Also note that we can only solve for the ratio $\frac{p_2}{p_1}$, not the absolute levels p_1 and p_2. This follows from the homogeneity of the aggregate excess demand function.

5(b) The Lagrangean for the planner is

$$L = a\left[\alpha \ln x_{A1} + (1-\alpha)\ln x_{A2}\right] + b\left[\beta \ln x_{B1} + (1-\beta)\ln x_{B2}\right]$$
$$- q_1\left[x_{A1} + x_{B1} - w_1\right] - q_2\left[x_{A2} + x_{B2} - w_2\right]$$

FOC: $\frac{a\alpha}{x_{A1}} = q_1, \quad \frac{a(1-\alpha)}{x_{A2}} = q_2$

$\frac{b\beta}{x_{B1}} = q_1, \quad \frac{b(1-\beta)}{x_{B2}} = q_2$

(we don't have to worry about SOC because the objective function is strictly concave)

From the FOC: $\underbrace{\frac{a\alpha}{x_{A1}} = \frac{b\beta}{x_{B1}}}_{\text{use } x_{B1} = w_1 - x_{A1}}$ and $\underbrace{\frac{a(1-\alpha)}{x_{A2}} = \frac{b(1-\beta)}{x_{B2}}}_{\text{use } x_{B2} = w_2 - x_{A2}}$

Making the substitutions, we obtain

$x_{A1} = \frac{a\alpha w_1}{a\alpha + b\beta} \qquad x_{A2} = \frac{a(1-\alpha)w_2}{a(1-\alpha) + b(1-\beta)}$

From the top line of the FOC (and some algebra)

$$\frac{q_2}{q_1} = \frac{\frac{a(1-\alpha)}{x_{A2}}}{\frac{a\alpha}{x_{A1}}} = \left(\frac{w_1}{w_2}\right)\frac{\left[a(1-\alpha) + b(1-\beta)\right]}{\left[a\alpha + b\beta\right]}$$

5(c) To get $\frac{p_2}{p_1} = \frac{q_2}{q_1}$ we need

$$\frac{(1-\alpha)w_{A1} + (1-\beta)w_{B1}}{\alpha w_{A2} + \beta w_{B2}} = \left(\frac{\omega_1}{\omega_2}\right)\frac{[a(1-\alpha) + b(1-\beta)]}{[a\alpha + b\beta]}$$

At this point, let's try for some intuition. We know $a > 0$ and $b > 0$ with $a + b = 1$. The interpretation of these weights is that they indicate how much the planner "cares about" each person. So we could try giving A the fraction a of both goods as an endowment and giving B the fraction b of both goods; that is,

$$\begin{cases} w_{A1} = a\,\omega_1 \\ w_{A2} = a\,\omega_2 \end{cases} \quad \text{and} \quad \begin{cases} w_{B1} = b\,\omega_1 \\ w_{B2} = b\,\omega_2 \end{cases}$$

If you plug these individual endowments into the equation at the top of the page, you will find that this works, and we get $\frac{p_2}{p_1} = \frac{q_2}{q_1}$.

In effect what we are doing is using the second welfare theorem. The social planner selects a particular Pareto efficient allocation based on the weights a and b. To support this allocation as a Walrasian equilibrium, we need prices that are equal to the Lagrange multipliers on the planner's feasibility constraints. If we are free to assign individual endowments, this can be accomplished.